

LOCAL LIMIT THEOREM FOR THE MAXIMUM OF A RANDOM WALK IN THE HEAVY-TRAFFIC REGIME

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ABSTRACT. Consider a family of Δ -lattice aperiodic random walks $\{S^{(a)}, 0 \leq a \leq a_0\}$ with increments $X_i^{(a)}$ and non-positive drift $-a$. Suppose that $\sup_{a \leq a_0} \mathbf{E}[(X^{(a)})^2] < \infty$ and $\sup_{a \leq a_0} \mathbf{E}[(\max\{0, X^{(a)}\})^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Assume that $X^{(a)} \xrightarrow{w} X^{(0)}$ as $a \rightarrow 0$ and denote by $M^{(a)} = \max_{k \geq 0} S_k^{(a)}$ the maximum of the random walk $S^{(a)}$. In this paper we provide the asymptotics of $\mathbf{P}(M^{(a)} = y\Delta)$ as $a \rightarrow 0$ in the case, when $y \rightarrow \infty$ and $ay = O(1)$. This asymptotics follows from a representation of $\mathbf{P}(M^{(a)} = y\Delta)$ via a geometric sum and a uniform renewal theorem, which is also proved in this paper.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\{S^{(a)}, a \in [0, a_0]\}$ denote a family of random walks with drift $-a \leq 0$ and increments $X_i^{(a)}$, that is,

$$S_0^{(a)} := 0, \quad S_n^{(a)} := \sum_{i=1}^n X_i^{(a)}, \quad n \geq 1.$$

We shall assume that $X_1^{(a)}, X_2^{(a)}, \dots$ are independent copies of a random variable $X^{(a)}$. In the case $a = 0$ we write S , X_i and X instead of $S^{(0)}$, $X_i^{(0)}$ and $X^{(0)}$ respectively. Assume that, as $a \rightarrow 0$,

$$X^{(a)} \xrightarrow{w} X \tag{1}$$

and

$$\sup_{a \in [0, a_0]} \mathbf{E}[X^{(a)}]^2 < \infty \quad \text{and} \quad \sup_{a \in [0, a_0]} \mathbf{E}[(\max\{0, X^{(a)}\})^{2+\varepsilon}] < \infty \tag{2}$$

for some $a_0, \varepsilon > 0$. If $a > 0$, the random walk $S^{(a)}$ drifts to $-\infty$ and the total maximum

$$M^{(a)} := \max_{k \geq 0} S_k^{(a)}$$

is finite almost surely. However, as $a \rightarrow 0$, $M^{(a)} \rightarrow \infty$ in probability. From this fact arises the natural question how fast $M^{(a)}$ grows as $a \rightarrow 0$. The first result concerning this question goes back to Kingman [7], who considered the case when $|X|$ has an exponential moment and proved that, as $a \rightarrow 0$,

$$\mathbf{P}(M^{(a)} > y) \sim e^{-2ay/\sigma^2} \tag{3}$$

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for all fixed values $y \geq 0$, where $\sigma^2 = \mathbf{Var}(X)$ denotes the variance of the increments in the case of zero-drift. Prokhorov [10] extended this result to the case that the increments have finite variance. Kingman and Prokhorov had a motivation for examining $M^{(a)}$ that comes from queueing theory: It is well known that a stationary distribution of the waiting time of a customer in a single-server first-come-first-served (GI/GI/1) queue coincides with that of the maximum of a corresponding random walk. In the context of queueing theory, the limit $a \rightarrow 0$ means that the traffic load tends to 1. Thus, the question on the distribution of $M^{(a)}$ may be seen as the question on the growth rate of a stationary waiting-time distribution in a GI/GI/1 queue. This is one of the most important questions in queueing theory and is usually referred to as heavy-traffic analysis.

Another interesting question is whether (3) remains valid, if we do not fix the value y , but consider $y = y(a) \rightarrow \infty$ as $a \rightarrow 0$ sufficiently slow. Olvera-Cravioto, Blanchet and Glynn [9] showed that, if the increments possess regular varying tails with index $r > 2$, there exists a critical value $y(a) \approx \sigma^2(r-2)a^{-1} \ln a^{-1}/2$, under which the heavy traffic approximation holds. Denisov and Kugler [5] (see also [2]) identified the critical value for general subexponential distributions, e.g. $y(a) \approx a^{-1/(1-\gamma)}$ in the Weibull case, where $\gamma \in (0, 1)$ is the parameter of the Weibull distribution.

In this paper we assume that $X^{(a)}$ possesses a Δ -lattice distribution, that means there exists some $\Delta > 0$ such that $\mathbf{P}(X \in \Delta\mathbb{Z}) = 1$ and Δ is the maximal positive number with this property. Let us assume without loss of generality that Δ is an integer. Our main result is a local limit theorem for the probability $\mathbf{P}(M^{(a)} = y\Delta)$ as $a \rightarrow 0$ for y such that $y \rightarrow \infty$ and $ay = O(1)$ under the assumption that the increments possess an aperiodic lattice distribution with zero-shift. The main idea for our proof is to find a representation of the probability $\mathbf{P}(M^{(a)} = y\Delta)$ as a geometric sum and to derive and apply a uniform renewal theorem to find the asymptotic behaviour of this sum. This uniform renewal theorem will be a generalization of a result attained by Nagaev [8].

It is worth mentioning that the approach used in this paper is similar to the method used in [2], where the authors use the well-known representation of $\mathbf{P}(M^{(a)} > y)$ as a geometric sum of independent random variables (see for example [1]) and a uniform renewal theorem from [3] to establish the asymptotic behaviour of $\mathbf{P}(M^{(a)} > y)$ as $a \rightarrow 0$ and $y \rightarrow \infty$ for subexponential distributions. In [3] there is also a uniform renewal theorem used to develop asymptotic expansions of the distribution of a geometric sum.

We now state our main result.

Theorem 1. *Assume that (1) and (2) hold and suppose that $X^{(a)}$ possesses an aperiodic Δ -lattice distribution for a small enough. Then, as $a \rightarrow 0$,*

$$\mathbf{P}(M^{(a)} = y\Delta) \sim \frac{2a\Delta}{\sigma^2} \exp\left\{-\frac{2ay\Delta}{\sigma^2}\right\} \quad (4)$$

uniformly for all y such that $y \rightarrow \infty$ and $ay = O(1)$ as $a \rightarrow 0$.

In the non-local case, it is known (see for example Wachtel and Shneer [12]) that one only needs to assume $\lim_{a \rightarrow 0} \mathbf{Var} X^{(a)} = \sigma^2 \in (0, \infty)$ and a Lindeberg-type condition

$$\lim_{a \rightarrow 0} \mathbf{E}[(X^{(a)})^2; |X_1^{(a)}| > K/a] = 0 \quad \text{for all } K > 0$$

to establish (3). This means that we must make stronger assumptions to establish our local result than it is needed in the non-local case.

Obviously, Theorem 1 restates the heavy traffic asymptotics (3): As $a \rightarrow \infty$,

$$\begin{aligned} \mathbf{P}(M^{(a)} \geq y\Delta) &= \sum_{x=y}^{\infty} \mathbf{P}(M^{(a)} = x\Delta) \sim \frac{2a\Delta}{\sigma^2} \sum_{x=y}^{\infty} e^{-2ax\Delta/\sigma^2} \\ &= \frac{2a\Delta}{\sigma^2} \frac{e^{-2ay\Delta/\sigma^2}}{1 - e^{-2ay\Delta/\sigma^2}} \sim e^{-2ay\Delta/\sigma^2} \end{aligned}$$

for all y such that $y \rightarrow \infty$ and $ay = O(1)$ as $a \rightarrow 0$.

2. UNIFORM RENEWAL THEOREM

In this section we prove a modification of Theorem 1 in Nagaev [8] which is, unlike the uniform renewal theorem from Nagaev, even uniform in the expected value. This renewal theorem is the key to the proof of our main result.

Consider a family of non-negative Δ -lattice and aperiodic random variables $\{Z^{(b)}, b \in I\}$ with $\mathbf{E}[Z^{(b)}] = b > 0$ and a non-empty set $I \subseteq \mathbb{R}$ that contains at least one accumulation point. Denote by $F^{(b)}$ the distribution function of $Z^{(b)}$ and by $F_k^{(b)}$ the k -fold convolution of $F^{(b)}$ with itself. Let

$$H(x, b, A) = \sum_{k=0}^{\infty} A^k F_k^{(b)}(x), \quad A > 0.$$

In renewal theory one usually studies the asymptotic behavior of $H(x+h, b, 1) - H(x, b, 1)$, $h > 0$. However, the case $A \neq 1$ is also of great interest. Nagaev's motivation for studying the case $A \neq 1$ comes from branching processes, since there arises a need for an asymptotic representation for $H(x+h, b, A) - H(x, b, A)$ as $x \rightarrow \infty$ with an estimate for the remainder term which is uniform in A . For our purposes we seek a representation for $H(x+h, b, A) - H(x, b, A)$ as $x \rightarrow \infty$ and the estimate for the remainder shall be uniform in A and b . Assume that there exists some $s > 1$ such that

$$\sup_{b \in I} \mathbf{E}[(Z^{(b)})^s] < \infty. \quad (5)$$

Put

$$\begin{aligned} f_{k\Delta}^{(b)} &= F^{(b)}(k\Delta) - F^{(b)}((k-1)\Delta), \quad f_y^{(b)}(z) = \sum_{k=0}^y f_{k\Delta}^{(b)} z^k, \\ \mu_y^{(b)}(z) &= f_y^{(b)'}(z) = \sum_{k=1}^y k f_{k\Delta}^{(b)} z^{k-1}. \end{aligned}$$

Proposition 2. *Let $\lambda_{y\Delta}^{(b)}(A)$ be the real non-negative root of the equation $A f_y^{(b)}(z) = 1$. Assume that (5) holds for some $s > 1$. Then, there exists a positive constant α for every accumulation point b_0 of I , such that*

$$\sum_{k=1}^{\infty} A^k \left(F_k^{(b)}(y\Delta) - F_k^{(b)}((y-1)\Delta) \right) = \frac{(\lambda_{y\Delta}^{(b)}(A))^{-y-1}}{A \mu_y^{(b)}(\lambda_{y\Delta}^{(b)}(A))} + o(y^{-\min\{1, s-1\}} \ln y) \quad (6)$$

uniformly in $b \in I \cap \{b \in I : |b - b_0| \leq \alpha\}$ and $A_y \leq A \leq 1$, where

$$A_y = 1 - C/y \quad (7)$$

with a fixed positive number C .

2.1. Proof of the uniform renewal theorem. Although the uniform renewal theorem is a generalization of Theorem 1 in Nagaev [8], the main idea of the proof is the same. However, for reasons of completeness, we give the whole proof.

Let us assume without loss of generality $\Delta = 1$, $I = [0, b_1]$ with $b_1 > 0$ and that y is sufficiently large in this section, even if it is not explicitly mentioned. Throughout the following $\int_a^b g(x) dF^{(b)}(x)$ is to be interpreted as $\int_{a+}^{b+} g(x) dF^{(b)}(x)$.

Lemma 3. *Assume that (5) holds for some I and $s > 1$. Put $\mu^{(b)} = \mathbf{E}[Z^{(b)}]$, $b \in I$, and $U_y(\delta) = \{z : 1 \leq |z| \leq e^{h_y}, |\arg z| \leq \delta\}$ for some $h_y = O(1/y)$. Then,*

$$\lim_{\delta \rightarrow 0} \lim_{y \rightarrow \infty} \sup_{b \in I, z \in U_y(\delta)} |\mu_y^{(b)}(z) - \mu^{(b)}| = 0. \quad (8)$$

Proof. First of all,

$$\begin{aligned} |\mu_y^{(b)}(z) - \mu^{(b)}| &= \left| \int_0^y x z^{x-1} dF^{(b)}(x) - \int_0^\infty x dF^{(b)}(x) \right| \\ &\leq \int_0^y x |z^{x-1} - 1| dF^{(b)}(x) + \int_y^\infty x dF^{(b)}(x). \end{aligned} \quad (9)$$

When $x, |z| \geq 1$, one can easily see by Taylor's approximation that

$$|z^{x-1} - 1| \leq x|z - 1||z|^x.$$

Using this estimate we obtain for all $z \in U_y(\delta)$ and $N \leq y$,

$$\begin{aligned} \int_0^N x |z^{x-1} - 1| dF^{(b)}(x) &\leq |z - 1| \int_0^N x^2 |z|^x dF^{(b)}(x) \\ &\leq |z - 1| e^{h_y y} \int_0^N x^2 dF^{(b)}(x) \leq N^2 |z - 1| e^{h_y y}. \end{aligned}$$

Further, a straightforward trigonometric calculation shows that for δ sufficiently small,

$$|z - 1| \leq |z - e^{i \arg z}| + |1 - e^{i \arg z}| = |z| - 1 + \sqrt{2(1 - \cos(\arg z))} \leq e^{h_y} - 1 + 2\delta$$

for all $z \in U_y(\delta)$ and hence, as $y \rightarrow \infty$,

$$\int_0^N x |z^{x-1} - 1| dF^{(b)}(x) \leq e^{h_y y} N^2 (e^{h_y} - 1 + 2\delta) = e^{h_y y} N^2 (2\delta + h_y + o(h_y))$$

uniformly in $b \in I$. At the same time, for $z \in U_y(\delta)$, assumption (5) and $h_y y = O(1)$ imply that there exists an absolute number $K > 0$ such that for all $N \leq y$,

$$\begin{aligned} \int_N^y x |z^{x-1} - 1| dF^{(b)}(x) &\leq (1 + e^{h_y y}) \int_N^y x dF^{(b)}(x) \\ &\leq \frac{1 + e^{h_y y}}{N^{s-1}} \int_N^\infty x^s dF^{(b)}(x) \leq K N^{1-s} \end{aligned}$$

and by setting $N = (2\delta + h_y)^{-1/3}$ and choosing K_1 such that $e^{h_y y} \leq K_1$ (which is possible due to the assumption $h_y = O(1/y)$), we attain

$$\begin{aligned} \int_0^y x |z^{x-1} - 1| dF^{(b)}(x) &\leq e^{h_y y} (2\delta + h_y)^{1/3} + K (2\delta + h_y)^{(s-1)/3} + o(h_y) \\ &\leq 2^{1/3} K_1 \delta^{1/3} + K 2^{(s-1)/3} \delta^{(s-1)/3} + o(1) \end{aligned} \quad (10)$$

uniformly in $b \in I$ as $y \rightarrow \infty$. Plugging the (10) into (9) and using (5) once more, we conclude

$$|\mu_y^{(b)}(z) - \mu^{(b)}| \leq 2^{1/3} K_1 \delta^{1/3} + K 2^{(s-1)/3} \delta^{(s-1)/3} + o(1) \quad (11)$$

uniformly in $b \in I$ as $y \rightarrow \infty$. \square

Lemma 4. *Assume that (5) holds for some I and $s > 1$. Then, for large enough y , $\lambda_y^{(b)}(A) < e^{h_y}$ for all $A_y \leq A \leq 1$ and $b \in I$, where $A_y = 1 - C/y$ with some constant $C > 0$ and $h_y = C_1/(\mu^{(0)}y)$ with $C_1 > C\mu^{(0)}/\inf_{b \in I} \mu^{(b)}$.*

Proof. We want to estimate the difference $\lambda_y^{(b)}(A) - 1$. First of all, by regarding the definition of $\lambda_y^{(b)}(A)$,

$$\begin{aligned} \int_{0-}^y \left((\lambda_y^{(b)}(A))^x - 1 \right) dF^{(b)}(x) &= f_y^{(b)}(\lambda_y^{(b)}(A)) - \int_{0-}^y dF^{(b)}(x) \\ &= \frac{1}{A} - 1 + \int_y^\infty dF^{(b)}(x) = \frac{1-A}{A} + \int_y^\infty dF^{(b)}(x). \end{aligned}$$

Further, $\lambda_y^{(b)}(A) \geq 1$ for $A \leq 1$ and therefore by the binomial formula,

$$(\lambda_y^{(b)}(A))^x - 1 \geq x(\lambda_y^{(b)}(A) - 1) \quad , x \geq 0.$$

Thus, uniformly in $b \in I$,

$$\begin{aligned} (\lambda_y^{(b)}(A) - 1) \int_{0-}^y x dF^{(b)}(x) &\leq \int_{0-}^y \left((\lambda_y^{(b)}(A))^x - 1 \right) dF^{(b)}(x) \\ &= \frac{1-A}{A} + \int_y^\infty dF^{(b)}(x) = \frac{1-A}{A} + O(y^{-s}), \end{aligned} \quad (12)$$

where we used (5) in the last line. The condition $A_y \leq A \leq 1$ implies that $1 - A \leq C/y$, hence

$$\frac{1}{A} = 1 + \frac{1-A}{A} = 1 + O\left(\frac{1}{y}\right)$$

and consequently

$$\frac{1-A}{A} \leq \frac{C}{Ay} = \frac{C}{y} + O\left(\frac{1}{y^2}\right). \quad (13)$$

From the inequalities (12), (13) and (5) we conclude that

$$\begin{aligned} \lambda_y^{(b)}(A) - 1 &\leq \frac{C/y + O(y^{-2}) + O(y^{-s})}{\mu^{(b)} - \int_y^\infty x dF^{(b)}(x)} = \frac{C/(\mu^{(b)}y)}{1 - O(y^{1-s})} + O(y^{-2}) + O(y^{-s}) \\ &= \frac{C}{\mu^{(b)}y} + O(y^{-2}) + O(y^{-s}) < \frac{C_1}{\mu^{(0)}y} \end{aligned}$$

uniformly in $b \in I$ for all y large enough. Therefore, since $e^x - 1 \geq x$ for all $x > 0$, $\lambda_y^{(b)}(A) < e^{h_y}$ uniformly in $A_y \leq A \leq 1$ and $b \in I$, if y is sufficiently large. \square

Lemma 5. *Assume that (5) holds for some I and $s > 1$. Put $h_y = C_1/(\mu^{(0)}y)$ with a constant $C_1 > C\mu^{(0)}/\inf_{b \in I} \mu^{(b)}$. Then, there exists some $b_2 > 0$ such that for y large enough, $A f_y^{(b)}(z) - 1$ has no other zeros in the disc $|z| < e^{h_y}$ apart from $\lambda_y^{(b)}(A)$ and this holds uniform in $A_y \leq A \leq 1$ and $0 \leq b \leq b_2$.*

Proof. First of all, for all $|z| \leq e^{h_y}$,

$$|\mu_y^{(b)}(z)| \leq \int_0^y x|z|^{x-1} dF^{(b)}(x) \leq e^{h_y y} \mu^{(b)}.$$

Using in addition $h_y y = O(1)$ and (5), we conclude

$$\sup_{y, b \leq b_1, |z| \leq e^{h_y}} |\mu_y^{(b)}(z)| < \infty. \quad (14)$$

Therefore,

$$\begin{aligned} & \lim_{y \rightarrow \infty} \sup_{b \leq b_1} \sup_{\substack{1 \leq r \leq e^{h_y} \\ 0 \leq \varphi \leq 2\pi}} \left| f_y^{(b)}(re^{i\varphi}) - f_y^{(b)}(e^{i\varphi}) \right| \\ &= \lim_{y \rightarrow \infty} \sup_{b \leq b_1} \sup_{\substack{1 \leq r \leq e^{h_y} \\ 0 \leq \varphi \leq 2\pi}} |\mu_y^{(b)}(e^{i\varphi})| |re^{i\varphi} - e^{i\varphi}| = 0. \end{aligned} \quad (15)$$

On the other hand,

$$\begin{aligned} \lim_{y \rightarrow \infty} \sup_{b \leq b_1} \sup_{0 \leq \varphi \leq 2\pi} |f_y^{(b)}(e^{i\varphi}) - f_\infty^{(b)}(e^{i\varphi})| &\leq \lim_{y \rightarrow \infty} \sup_{b \leq b_1} \sup_{0 \leq \varphi \leq 2\pi} \int_y^\infty |e^{i\varphi x}| dF^{(b)}(x) \\ &= \lim_{y \rightarrow \infty} \sup_{b \leq b_1} \overline{F}^{(b)}(y) = 0. \end{aligned} \quad (16)$$

As $b \rightarrow 0$, $F^{(b)}(\cdot) \rightarrow F^{(0)}(\cdot)$ in the sense of Definition 3 from chapter VIII.1 in Feller [6] and $F^{(0)}$ is not defective because of (5). Obviously, $u_\varphi(\cdot) = e^{i\varphi \cdot}$ is equicontinuous with $|u_\varphi| = 1 < \infty$. Hence, by a corollary in chapter VIII.1 in Feller [6],

$$\int_0^\infty e^{i\varphi x} dF^{(b)}(x) \rightarrow \int_0^\infty e^{i\varphi x} dF^{(0)}(x) \quad (17)$$

uniformly in $0 \leq \varphi \leq \pi$ as $b \rightarrow 0$.

Now, let us first consider values of z in the circle $|z| < e^{h_y}$ that are not in the vicinity of $\lambda_y^{(b)}(A)$. Due to Lemma 4, these values can be characterized as those values that satisfy $|z| < e^{h_y}$ and $\delta \leq |\arg z| \leq \pi$, $\delta > 0$. It is

$$\sup_{\delta \leq \varphi \leq \pi} |f_\infty^{(0)}(e^{i\varphi})| = \sup_{\delta \leq \varphi \leq \pi} \left| \int_0^\infty e^{i\varphi x} dF^{(0)}(x) \right| < \sup_{\delta \leq \varphi \leq \pi} \int_0^\infty |e^{i\varphi x}| dF^{(0)}(x) = 1.$$

Combining the latter inequality with (17), we conclude that there exists some $b_2 > 0$ (assume without loss of generality $b_2 \leq b_1$) such that

$$\sup_{b \leq b_2} \sup_{\delta \leq \varphi \leq \pi} |f_\infty^{(b)}(e^{i\varphi})| < 1$$

and since this inequality is strict,

$$m(\delta) := \inf_{b \leq b_2} \inf_{A \leq 1} \inf_{\delta \leq \varphi \leq \pi} |Af_\infty^{(b)}(e^{i\varphi}) - 1| > 0. \quad (18)$$

By combining (15), (16) and (18), we conclude that for large enough y and $A \in \mathfrak{A}_y$,

$$\inf_{b \leq b_2} \inf_{\substack{1 \leq r \leq e^{h_y} \\ \delta \leq \varphi \leq 2\pi}} |Af_y^{(b)}(re^{i\varphi}) - 1| > \frac{m(\delta)}{2} > 0. \quad (19)$$

On the basis of (19) we can assert that if $Af_y^{(b)}(z) - 1$ has a zero $\tilde{\lambda}_y^{(b)}(A)$ in the disc $|z| \leq e^{h_y}$ differing from $\lambda_y^{(b)}(A)$, then $\tilde{\lambda}_y^{(b)}(A)$ will lie outside the region $\{z : 1 \leq |z| \leq e^{h_y}, |\arg z| \geq \delta\}$ and this holds uniformly in $b \in [0, b_2]$ and $A \in \mathfrak{A}_y$.

Next, consider the region $U_y(\delta) = \{z : 1 \leq |z| \leq e^{h_y}, |\arg z| < \delta\}$. Observe that Taylor's formula implies

$$Af_y^{(b)}(z) - 1 = Af_y^{(b)}(z) - Af_y^{(b)}(\lambda_y^{(b)}(A)) \geq A\mu_y^{(b)}(\lambda_y^{(b)}(A))(z - \lambda_y^{(b)}(A)).$$

This inequality plus the equicontinuity of $f_y^{(b)}(z)$ imply the existence of a $\delta_1(b, A) > 0$ such that $|Af_y^{(b)}(z) - 1|$ has no other zeros in the disc $|z - \lambda_y^{(b)}(A)| \leq \delta_1(b, A)$ apart from $\lambda_y^{(b)}(A)$. Therefore,

$$\tilde{m}(\delta_2) := \inf_{b \leq b_2} \inf_{A \in \mathfrak{A}_y} \inf_{z: |z - \lambda_y^{(b)}(A)| \leq \delta_2} |Af_y^{(b)}(z) - 1| > 0.$$

where $\delta_2 = \inf_{b \leq b_2} \inf_{A \in \mathfrak{A}_y} \delta_1(b, A) > 0$. Observe that $\lambda_y^{(b)}(A) \geq 1$ for $A \leq 1$ and $\lambda_y^{(b)}(A) < e^{h_y}$ by Lemma 4. Hence, for δ small enough, say $\delta \leq \delta_3$, the region

$$\bigcup_{b \leq b_2} \bigcup_{A \in \mathfrak{A}_y} \{z : |z - \lambda_y^{(b)}(A)| \leq \delta_1\}$$

covers $U_y(\delta)$ and that means $\tilde{\lambda}_y^{(b)}(A)$ cannot lie in the region $\{z : 1 - \varepsilon_0 \leq |z| \leq e^{h_y}, |\arg z| < \delta_3\}$. Setting $\delta = \delta_3$ in (19) we conclude that $\tilde{\lambda}_y^{(b)}(A)$ cannot lie in the annulus $1 \leq |z| \leq e^{h_y}$. Since $|\tilde{\lambda}_y^{(b)}(A)| \geq 1$ for all $A \leq 1$, we finally obtain that $\tilde{\lambda}_y^{(b)}(A)$ does not lie in the disc $|z| \leq e^{h_y}$, so $\lambda_y^{(b)}(A)$ is the only root of the equation $Af_y^{(b)}(A) = 1$ in the disc $|z| \leq e^{h_y}$ and this holds uniformly in $b \leq b_2$ and $A_y \leq A \leq 1$. \square

Proof of Proposition 2. Let γ_y be a circle of radius $r_y = e^{h_y}$ with $h_y = C_1/(\mu^{(0)}y)$, $C_1 > \mu^{(0)} + C\mu^{(0)}/\inf_{b \leq b_1} \mu^{(b)}$ and C from (7). Then, according to Lemma 4 and Lemma 5, there exists some $b_2 > 0$ such that for all $0 \leq b \leq b_2$ and $A \in \mathfrak{A}_y$, the function $1 - Af_y^{(b)}(z)$ is zero in the disc $|z| \leq e^{h_y}$, if and only if $z = \lambda_y^{(b)}(A)$. Hence, the Residue theorem states that

$$\frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz = \text{Res} \left(\frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, \lambda_y^{(b)}(A) \right) + \text{Res} \left(\frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, 0 \right). \quad (20)$$

for $0 \leq b \leq b_2$ and $A \in \mathfrak{A}_y$.

In the following denote by $C_n(f(z))$, $n \geq 1$, the coefficient of z^n in the Taylor series of the function $f(z)$. An easy calculation shows that

$$A^n(f_\infty^{(b)}(z))^n = A^n \sum_{j=1}^{\infty} \left(F_n^{(b)}(j) - F_n^{(b)}(j-1) \right) z^j$$

and consequently, by changing the order of summation, it is not hard to see that

$$\sum_{k=1}^{\infty} A^k \left(F_k^{(b)}(n) - F_k^{(b)}(n-1) \right) = C_n \left(\frac{1}{1 - Af_\infty^{(b)}(z)} \right).$$

On the other hand, when $n \leq y$,

$$C_n \left(\frac{1}{1 - Af_\infty^{(b)}(z)} \right) = C_n \left(\frac{1}{1 - Af_y^{(b)}(z)} \right)$$

and thus, for $n \leq y$,

$$\sum_{k=1}^{\infty} A^k \left(F_k^{(b)}(n) - F_k^{(b)}(n-1) \right) = C_n \left(\frac{1}{1 - Af_y^{(b)}(z)} \right). \quad (21)$$

Regarding (21) with $n = y$, one can easily verify

$$\text{Res} \left(\frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, 0 \right) = \sum_{k=1}^{\infty} A^k \left(F_k^{(b)}(y) - F_k^{(b)}(y-1) \right).$$

The pole of the function $z^{-y-1}/(1 - Af_y^{(b)}(z))$ in $z = \lambda_y^{(b)}(A)$ is of order 1. Therefore, it is not hard to see that

$$\text{Res} \left(\frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, \lambda_y^{(b)}(A) \right) = -\frac{\lambda_y^{(b)}(A)^{-y-1}}{A\mu_y^{(b)}(\lambda_y^{(b)}(A))}$$

and by combining the latter results we obtain

$$\sum_{k=1}^{\infty} A^k \left(F_k^{(b)}(y) - F_k^{(b)}(y-1) \right) = \frac{(\lambda_y^{(b)}(A))^{-y-1}}{A\mu_y^{(b)}(\lambda_y^{(b)}(A))} + \frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz.$$

It remains to show that under the conditions of Proposition 2,

$$\frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz = o \left(y^{-\min\{1, s-1\}} \ln y \right) \quad (22)$$

uniformly in $b \leq b_2$ and $A_y \leq A \leq 1$. Let

$$\begin{aligned} \varphi_y^{(b)}(z) &= A(f_y^{(b)}(z) - f_y^{(b)}(r_y)) - A\mu_y^{(b)}(r_y)(z - r_y), \\ \psi_y^{(b)}(z) &= 1 - Af_y^{(b)}(r_y) - A\mu_y^{(b)}(r_y)(z - r_y). \end{aligned}$$

Then, the following identity holds:

$$\frac{1}{1 - Af_y^{(b)}(z)} - \frac{1}{\psi_y^{(b)}(z)} = \frac{\psi_y^{(b)}(z) - 1 + Af_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} = \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)}. \quad (23)$$

Let $\varepsilon > 0$, $\gamma_y(\varepsilon) = \gamma_y \cap U_y(\varepsilon)$ and let $\bar{\gamma}_y(\varepsilon)$ be the complement of $\gamma_y(\varepsilon)$ with respect to γ_y . By (23),

$$\begin{aligned} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz &= \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz + \int_{\gamma_y(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} dz \\ &\quad + \int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} dz. \end{aligned}$$

Using (23) once again, the last integral of the latter identity can be rewritten as

$$\int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} dz = - \int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz + \int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz.$$

Hence,

$$\int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz = I_1^{(b)}(y) + \sum_{j=2}^4 I_j^{(b)}(y, \varepsilon), \quad (24)$$

where

$$\begin{aligned} I_1^{(b)}(y) &= \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz, & I_2^{(b)}(y, \varepsilon) &= \int_{\gamma_y(\varepsilon)} \frac{z^{-y-1} \varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z)) \psi_y^{(b)}(z)} dz, \\ I_3^{(b)}(y, \varepsilon) &= - \int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz, & I_4^{(b)}(y, \varepsilon) &= \int_{\bar{\gamma}_y(\varepsilon)} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz. \end{aligned}$$

To calculate $I_1^{(b)}$ let us examine integrals of the form

$$\int_{|z|=c^2} \frac{z^{-n}}{dz+h} dz, \quad (25)$$

where $n > 0$, $d, h \in \mathbb{C}$ and $|h| < c^2|d|$. For $|h| < c^2|d|$, the function $z^{-n}/(dz+h)$ has exactly two singularities in the disc $|z| \leq c^2$, one in 0 and the other in $-h/d$. Consequently the Residue theorem states that

$$\int_{|z|=c^2} \frac{z^{-n}}{dz+h} dz = \text{Res} \left(\frac{z^{-n}}{dz+h}, 0 \right) + \text{Res} \left(\frac{z^{-n}}{dz+h}, -\frac{h}{d} \right).$$

The pole in $z = 0$ has order n , hence

$$\text{Res} \left(\frac{z^{-n}}{dz+h}, 0 \right) = (-1)^{n-1} d^{n-1} h^{-n}$$

and the pole in $z = -h/d$ is of order 1, thus

$$\text{Res} \left(\frac{z^{-n}}{dz+h}, -\frac{h}{d} \right) = (-1)^n d^{n-1} h^{-n}.$$

Therefore,

$$\int_{|z|=c^2} \frac{z^{-n}}{dz+h} dz = [(-1)^{n-1} + (-1)^n] d^{n-1} h^{-n} = 0. \quad (26)$$

By the equicontinuity of $\mu_y^{(b)}(\cdot)$, the result from (14), Lemma 3 and Lemma 4, as $y \rightarrow \infty$,

$$\begin{aligned} f_y^{(b)}(r_y) - f_y^{(b)}(\lambda_y^{(b)}(A)) &= (r_y - \lambda_y^{(b)}(A)) \mu^{(b)}(\lambda_y^{(b)}(A)) + o(r_y - \lambda_y^{(b)}(A)) \\ &= (r_y - \lambda_y^{(b)}(A)) \mu^{(b)} + o(r_y - \lambda_y^{(b)}(A)) \end{aligned} \quad (27)$$

uniformly in $b \leq b_2$ and $A \in \mathfrak{A}_y$. By virtue of Lemma 4 and the definition of C_1 , $|\lambda_y^{(b)}(A)| \leq e^{h_y-1/y}$ and consequently

$$\begin{aligned} r_y - \lambda_y^{(b)}(A) &\geq e^{h_y}(1 - e^{-1/y}) \\ &= (1 + h_y + o(y^{-1}))(y^{-1} + o(y^{-1})) = y^{-1} + o(y^{-1}) \end{aligned}$$

uniformly in $b \leq b_2$ and $A \in \mathfrak{A}_y$. By plugging this results into (27),

$$1 - Af_y^{(b)}(r_y) \leq -\frac{A\mu^{(b)}}{y} + o\left(\frac{1}{y}\right) < 0 \quad (28)$$

for y large enough. Now put $h = 1 - Af_y^{(b)}(r_y) + A\mu_y^{(b)}(r_y)r_y$ and $d = -A\mu_y^{(b)}(r_y)$. Then, since $A\mu_y^{(b)}(r_y)r_y \geq A\mu_y^{(b)}(1) \neq o(1)$, we obtain by virtue of (28),

$$|h| \leq A\mu_y^{(b)}(r_y)r_y = |d|r_y$$

and consequently by (26),

$$I_1^{(b)}(y) = \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz = 0. \quad (29)$$

Let us now consider $I_2^{(b)}$. Clearly,

$$I_2^{(b)}(y, \varepsilon) = ir_y^{-y} \int_{\varepsilon}^{2\pi-\varepsilon} \frac{\varphi_y^{(b)}(r_y e^{it})}{(1 - Af_y^{(b)}(r_y e^{it}))\psi_y^{(b)}(r_y e^{it})} e^{-ity} dt.$$

Taiblesons [11] estimate for Fourier coefficients states that for any function f with bounded variation on $[0, 2\pi]$ and $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ as $x \rightarrow \infty$, it is

$$|c_n| \leq \frac{2\pi \text{var}(f)}{n},$$

where var denotes the variation of f , defining this to be the sum of the variations of the real and the imaginary parts. Hence,

$$I_2^{(b)}(y, \varepsilon) = O\left(\frac{1}{y} \text{var}_{z \in \gamma_y(\varepsilon)} \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)}\right). \quad (30)$$

The variation of $\omega_y^{(b)}(z) := \varphi_y^{(b)}(z)/((1 - Af_y^{(b)}(z))\psi_y^{(b)}(z))$ on $\gamma_y(\varepsilon)$ can be rewritten as follows:

$$\begin{aligned} \text{var}_{z \in \gamma_y(\varepsilon)} \omega_y^{(b)}(z) &= \text{var}_{z \in \gamma_y(\varepsilon)} \text{Re}(\omega_y^{(b)}(z)) + \text{var}_{z \in \gamma_y(\varepsilon)} \text{Im}(\omega_y^{(b)}(z)) \\ &= \int_{\gamma_y(\varepsilon)} \left(\left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right| + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right| \right) dl, \end{aligned}$$

where dl is the differential of the arc along $\gamma_y(\varepsilon)$. Due to the binomial formula,

$$\begin{aligned} \left(\left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right| + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right| \right)^2 &\leq 2 \left(\left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right|^2 + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right|^2 \right) \\ &= 2 \left| \frac{d}{dz} \omega_y^{(b)}(z) \right|^2 \end{aligned}$$

and thus,

$$\begin{aligned} \text{var}_{z \in \gamma_y(\varepsilon)} \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} &\leq \sqrt{2} \int_{\gamma_y(\varepsilon)} \left| \frac{d}{dz} \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} \right| dz \\ &\leq \sqrt{2} \left(\int_{\gamma_y(\varepsilon)} \left| \frac{\psi_y^{(b)'}(z)\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))(\psi_y^{(b)}(z))^2} \right| dz + \int_{\gamma_y(\varepsilon)} \left| \frac{A\mu_y^{(b)}(z)\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))^2\psi_y^{(b)}(z)} \right| dz \right. \\ &\quad \left. + \int_{\gamma_y(\varepsilon)} \left| \frac{\varphi_y^{(b)'}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} \right| dz \right) \\ &= \sqrt{2}(I_{21}^{(b)} + I_{22}^{(b)} + I_{23}^{(b)}). \end{aligned} \quad (31)$$

Let us bound the terms appearing in the integrands of the integrals from the latter inequality. Using the definition of the complex absolute value, an easy calculation

shows that

$$|Af_y^{(b)}(z) - 1|^2 = A^2|f_y^{(b)}(z) - f_y^{(b)}(r_y)|^2 + |Af_y^{(b)}(r_y) - 1|^2 - 2A(Af_y^{(b)}(r_y) - 1)\operatorname{Re}(f_y^{(b)}(r_y) - f_y^{(b)}(z)).$$

By the equicontinuity and Lemma 3, as $y \rightarrow \infty$,

$$|f_y^{(b)}(r_y) - f_y^{(b)}(z)| = |r_y - z|\mu^{(b)}(z) + o(r_y - z) \geq (1 - \delta)\mu^{(b)}|z - r_y| \quad (32)$$

and

$$|f_y^{(b)}(r_y) - f_y^{(b)}(z)| = |r_y - z|\mu^{(b)}(z) + o(r_y - z) \leq (1 + \delta)\mu^{(b)}|z - r_y| \quad (33)$$

uniformly in $b \leq b_2$ and $z \in U_y^{(b)}(\delta)$, if δ is small enough. Further, for all $z \in U_y(\delta)$ with δ sufficiently small,

$$\operatorname{Re}(r_y - z) = \sin(\arg z)|z - r_y| \leq \delta|z - r_y|.$$

By the virtue of (28), (32) and (33),

$$|Af_y^{(b)}(z) - 1|^2 \geq |1 - Af_y^{(b)}(r_y)|^2 + (1 - \delta)(\mu^{(b)})^2 A^2 |z - r_y|^2 - 2\delta(1 + \delta)\mu^{(b)} A(Af_y^{(b)}(r_y) - 1)|z - r_y|$$

and by the binomial formula for δ sufficiently small,

$$2\mu^{(b)} A(Af_y^{(b)}(r_y) - 1)|z - r_y| \leq (\mu^{(b)})^2 A^2 |z - r_y|^2 + |1 - Af_y^{(b)}(r_y)|^2.$$

Therefore, again by the binomial formula,

$$\begin{aligned} |Af_y^{(b)}(z) - 1|^2 &\geq (1 - \delta - \delta(1 + \delta)) \left[|1 - Af_y^{(b)}(r_y)|^2 + (\mu^{(b)})^2 A^2 |z - r_y|^2 \right] \\ &\geq \frac{1 - \delta - \delta(1 + \delta)}{2} \left[|1 - Af_y^{(b)}(r_y)| + A\mu^{(b)}|z - r_y| \right]^2. \end{aligned}$$

Since δ can be chosen arbitrary small, one can especially choose δ so small that $1 - \delta - \delta(1 + \delta) \geq 1/2$. Thus,

$$|Af_y^{(b)}(z) - 1| \geq \frac{|Af_y^{(b)}(r_y) - 1|}{2} + \frac{A\mu^{(b)}|z - r_y|}{2} \quad (34)$$

uniformly in $b \leq b_2$ and $z \in U_y^{(b)}(\delta)$. We proceed analogously to bound $|\psi_y^{(b)}(z)|$ for $z \in U_y(\delta)$ from below. It is $\operatorname{Re}(r_y - z) \leq |z - r_y|$ and by virtue of Lemma 3, $\mu_y^{(b)}(r_y) \in [(1 - \hat{\delta}_1)\mu^{(b)}, (1 + \hat{\delta}_1)\mu^{(b)}]$ for y large enough. Consequently, one can easily see that for $\hat{\delta}_1$ small enough,

$$\begin{aligned} |\psi_y^{(b)}(z)|^2 &= |1 - f_y^{(b)}(r_y)|^2 + A^2 \left(\mu_y^{(b)}(r_y) \right)^2 |z - r_y|^2 \\ &\quad - 2A(f_y^{(b)}(r_y) - 1)\mu_y^{(b)}(r_y)\operatorname{Re}(r_y - z) \\ &\geq \frac{1 - \hat{\delta}_2}{2} \left[|1 - f_y^{(b)}(r_y)| + A\mu^{(b)}|z - r_y| \right]^2 \end{aligned} \quad (35)$$

for all $\hat{\delta}_2 \leq 1/2$. Hence,

$$|\psi_y^{(b)}(z)| \geq \frac{|1 - Af_y^{(b)}(r_y)|}{4} + \frac{A\mu^{(b)}|z - r_y|}{4}. \quad (36)$$

On the other hand, one can easily see that for every z on $\gamma_y(\varepsilon)$ with ε sufficiently small,

$$\begin{aligned} |z - r_y| &\geq |e^{i \arg z} - 1| = \sqrt{\sin^2(\arg z) + (1 - \cos(\arg z))^2} \\ &= \sqrt{2 - 2 \cos(\arg z)} \geq \frac{|\arg z|}{2}, \end{aligned} \quad (37)$$

where we used $\cos \varphi \leq 1 - \varphi^2/8$ in the last inequality. Combining inequalities (28) and (37) with (34), we obtain

$$|1 - Af_y^{(b)}(z)| \geq \frac{A\mu^{(b)}}{4} \left(\frac{1}{y} + |\arg z| \right). \quad (38)$$

for $b \leq b_2$ and $z \in U_y^{(b)}(\delta)$. The inequalities (28), (37) and (36) provide

$$|\psi_y^{(b)}(z)| \geq \frac{A\mu^{(b)}}{8} \left(\frac{1}{y} + |\arg z| \right) \quad (39)$$

and, moreover, an easy calculation shows

$$|\psi_y^{(b)'}(z)| = A\mu_y^{(b)}(r_y) \leq e^{h_y y} A\mu^{(b)}. \quad (40)$$

For $z \in U_y(\delta)$,

$$|f_{y1}^{(b)''}(z)| \leq \begin{cases} e^{h_y y} \mathbf{E}(Z^{(b)})^2 & : s \geq 2 \\ e^{h_y y} y^{2-s} \mathbf{E}(Z^{(b)})^s & : 1 < s < 2 \end{cases}$$

and consequently,

$$\frac{\varphi_y^{(b)}(z)}{|z - r_y|^2 y^{\max\{0, 2-s\}}} \sim \frac{\varphi_y^{(b)'}(z)}{2|z - r_y| y^{\max\{0, 2-s\}}} \sim \frac{Af_y^{(b)''}(z)}{2y^{\max\{0, 2-s\}}} = O(1)$$

as $y \rightarrow \infty$. By virtue of (37),

$$|e^{i \arg z}| = \sqrt{2 - 2 \cos(\arg z)} \leq \arg z,$$

if $\arg z$ is sufficiently small. Hence, if $\arg z$ is sufficiently small,

$$\varphi_y^{(b)}(z) = O(y^{\max\{0, 2-s\}} |z - r_y|^2) = O(y^{\max\{0, 2-s\}} \arg^2(z)) \quad (41)$$

and

$$\varphi_y^{(b)'}(z) = O(y^{\max\{0, 2-s\}} |z - r_y|) = O(y^{\max\{0, 2-s\}} |\arg(z)|) \quad (42)$$

uniformly in $b \leq b_2$ and $A \in \mathfrak{A}_y$. Considering (38), (39), (40), (41) and $h_y y = O(1)$ provides

$$|I_{21}^{(b)}| \leq r_y \int_{-\varepsilon}^{\varepsilon} \frac{|\psi_y^{(b)'}(r_y e^{it})| |\varphi_y^{(b)}(r_y e^{it})|}{|f_y^{(b)}(r_y e^{it}) - 1| |\psi_y^{(b)}(r_y e^{it})|^2} dt = O \left(y^{\max\{0, 2-s\}} \int_0^{\varepsilon} \frac{t^2}{(y^{-1} + t)^3} dt \right)$$

uniformly in $b \leq b_2$ and $A \in \mathfrak{A}_y$. Moreover,

$$\begin{aligned} \int_0^{\varepsilon} \frac{t^2}{(y^{-1} + t)^3} dt &= \int_{1/y}^{\varepsilon+1/y} \frac{(w - y^{-1})^2}{w^3} dw \\ &\sim \ln(\varepsilon + y^{-1}) - \ln(y^{-1}) = \ln(1 + \varepsilon y) \sim \ln(y) \end{aligned} \quad (43)$$

and therefore, uniformly in $b \leq b_2$ and $A \in \mathfrak{A}_y$,

$$|I_{21}^{(b)}| = O(y^{\max\{0, 2-s\}} \ln y). \quad (44)$$

In analogy, by additionally taking into account that $\mu_y^{(b)}(z) \leq 2\mu^{(b)}$ due to Lemma 3 for y large enough, one can easily see that

$$I_{22}^{(b)} = O(y^{\max\{0, 2-s\}} \ln y) \quad (45)$$

and furthermore, by regarding (42),

$$I_{23}^{(b)} = O\left(y^{\max\{0, 2-s\}} \int_0^\varepsilon \frac{t}{(y^{-1} + t)^2} dt\right) = O(y^{\max\{0, 2-s\}} \ln y). \quad (46)$$

Finally, plugging (44), (45) and (46) into (31) we attain

$$\text{var}_{z \in \overline{\gamma}_y(\varepsilon)} \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))\psi_y^{(b)}(z)} = O(y^{\max\{0, 2-s\}} \ln y)$$

and hence by (30),

$$|I_2^{(b)}(y, \varepsilon)| = o(y^{\max\{-1, -(s-1)\}} \ln y) \quad (47)$$

uniformly in $b \leq b_2$ and the admissible values of A . Next, we draw our attention to the integral $I_3^{(b)}$.

$$I_3^{(b)}(y, \varepsilon) = -ir_y^{-y} \int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{\psi_y^{(b)}(r_y e^{it})} dt. \quad (48)$$

To bound this integral we use Taibleson's estimate for Fourier coefficients again:

$$\int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{\psi_y^{(b)}(r_y e^{it})} dt = O\left(\frac{1}{y} \text{var}_{z \in \overline{\gamma}_y(\varepsilon)} \frac{1}{\psi_y^{(b)}(z)}\right) \quad (49)$$

In analogy to (31), one can show that

$$\text{var}_{z \in \overline{\gamma}_y(\varepsilon)} \frac{1}{\psi_y^{(b)}(z)} \leq \sqrt{2} \int_{\overline{\gamma}_y(\varepsilon)} \left| \frac{d}{dz} \frac{1}{\psi_y^{(b)}(z)} \right| dz \leq \sqrt{2} \int_{\overline{\gamma}_y(\varepsilon)} \frac{|\psi_y^{(b)'}(z)|}{|\psi_y^{(b)}(z)|^2} dz.$$

By (36),

$$|\psi_y^{(b)}(z)|^2 \geq \frac{A^2(\mu^{(b)})^2}{16} |z - r_y|^2 \geq \frac{A^2(\mu^{(b)})^2}{16} \varepsilon^2,$$

where we used that $|z - r_y| > \varepsilon$ for all $z \in \overline{\gamma}_y(\varepsilon)$. Therefore, by (40),

$$\text{var}_{z \in \overline{\gamma}_y(\varepsilon)} \frac{1}{\psi_y^{(b)}(z)} = O(1)$$

and consequently by combining this result with (48), (49) and $h_y y = O(1)$,

$$|I_3^{(b)}(y, \varepsilon)| = O\left(\frac{1}{y}\right) \quad (50)$$

uniform in $b \leq b_2$ and $A \in \mathfrak{A}_y$. It remains to consider $I_4^{(b)}$.

$$|I_4^{(b)}(y, \varepsilon)| = \left| ir_y^{-y} \int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{1 - Af_y^{(b)}(r_y e^{it})} dt \right| = O\left(\frac{1}{y} \text{var}_{z \in \overline{\gamma}_y(\varepsilon)} \frac{1}{1 - Af_y^{(b)}(z)}\right). \quad (51)$$

Further, by (14) and (19),

$$\begin{aligned} \text{var}_{z \in \bar{\gamma}_y(\varepsilon)} \frac{1}{1 - Af_y^{(b)}(z)} &\leq \sqrt{2} \int_{\bar{\gamma}_y(\varepsilon)} \left| \frac{d}{dz} \frac{1}{1 - Af_y^{(b)}(z)} \right| dz \\ &= \sqrt{2} \int_{\bar{\gamma}_y(\varepsilon)} \frac{|\mu_y^{(b)}(z)|}{|1 - Af_y^{(b)}(z)|^2} dz = O(1) \end{aligned}$$

and consequently

$$I_4^{(b)}(y, \varepsilon) = O\left(\frac{1}{y}\right). \quad (52)$$

Finally, by plugging the results attained in (29), (47), (50) and (52) into (24), we get

$$\begin{aligned} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} dz &= o(y^{\max\{-1, -(s-1)\}} \ln y) + o(y^{-(s-1)}) + O(y^{-1}) \\ &= o(y^{-\min\{1, s-1\}} \ln y) \end{aligned} \quad (53)$$

uniformly in $0 \leq b \leq b_2$ and $A_y \leq A \leq 1$.

3. PROOF OF THE LOCAL LIMIT THEOREM

Put $\tau_{+,0}^{(a)} = 0$ and define recursively for $i \geq 1$ the i -th strict ascending ladder epoch of the random walk $S^{(a)}$ and its corresponding ladder height by

$$\tau_{+,i}^{(a)} := \min\{k \geq \tau_{+,i-1}^{(a)} : S_k^{(a)} > S_{\tau_{+,i-1}^{(a)}}^{(a)}\} \quad \text{and} \quad \chi_i^{(a)} = S_{\tau_{+,i}^{(a)}}^{(a)} - S_{\tau_{+,i-1}^{(a)}}^{(a)}.$$

In the case $i = 1$ we write $\tau_+^{(a)}$ and $\chi^{(a)}$ instead of $\tau_{+,1}^{(a)}$ and $\chi_1^{(a)}$ respectively and, if additionally $a = 0$, we write τ_+ and χ instead of $\tau_+^{(0)}$ and $\chi^{(0)}$ respectively. Define random variables $Z_i^{(a)}$ as *iid* copies of a random variable $Z^{(a)}$ with

$$\mathbf{P}(Z^{(a)} \in \cdot) = \mathbf{P}(\chi_1^{(a)} \in \cdot | \tau_+^{(a)} < \infty).$$

Denote by $\theta := \min\{k \geq 0 : S_k^{(a)} = M^{(a)}\}$ the first time the random walk reaches its maximum. Then,

$$\mathbf{P}(M^{(a)} = y\Delta) = \sum_{n=1}^{\infty} \mathbf{P}(M^{(a)} = y\Delta, \theta = n).$$

We further define $M_n^{(a)} := \max_{k \leq n} S_k^{(a)}$ and $\theta_n := \min\{k \leq n : S_k^{(a)} = M_n^{(a)}\}$. By the Markov property,

$$\mathbf{P}(M^{(a)} = y\Delta, \theta = n) = \mathbf{P}(S_n^{(a)} = y\Delta, \theta_n = n) \mathbf{P}(\tau_a^+ = \infty).$$

Hence the following representation holds for the maximum:

$$\mathbf{P}(M^{(a)} = y\Delta) = \mathbf{P}(\tau_+^{(a)} = \infty) \sum_{n=1}^{\infty} \mathbf{P}(S_n^{(a)} = y\Delta, \theta_n = n). \quad (54)$$

Clearly,

$$\begin{aligned} \mathbf{P}(S_n^{(a)} = y\Delta, \theta_n = n) &= \mathbf{P}(S_n^{(a)} = y\Delta, n \text{ is a strict ascending ladder epoch}) \\ &= \sum_{k=1}^{\infty} \mathbf{P}(\chi_1^{(a)} + \chi_2^{(a)} + \cdots + \chi_k^{(a)} = y\Delta, \tau_{+,1}^{(a)} + \tau_{+,2}^{(a)} + \cdots + \tau_{+,k}^{(a)} = n). \end{aligned} \quad (55)$$

Denote the distribution function of $Z^{(a)}$ by $F^{(\mu^{(a)})}$, where $\mu^{(a)} = \mathbf{E}[Z^{(a)}]$, and let $F_k^{(\mu^{(a)})}$ be the k -fold convolution of $F^{(\mu^{(a)})}$ with itself. Then, by using (55), changing the order of summation and using the Markov property,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \mathbf{P}(S_n^{(a)} = y\Delta, \theta_n = n) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mathbf{P}(\chi_1^{(a)} + \chi_2^{(a)} + \cdots + \chi_k^{(a)} = y\Delta, \tau_{+,1}^{(a)} + \tau_{+,2}^{(a)} + \cdots + \tau_{+,k}^{(a)} = n) \\
&= \sum_{k=1}^{\infty} \mathbf{P}(\chi_1^{(a)} + \chi_2^{(a)} + \cdots + \chi_k^{(a)} = y\Delta | \tau_{+,k}^{(a)} < \infty) \mathbf{P}(\tau_{+,k}^{(a)} < \infty) \\
&= \sum_{k=1}^{\infty} A^k \left(F_k^{(\mu^{(a)})}(y\Delta) - F_k^{(\mu^{(a)})}((y-1)\Delta) \right) \tag{56}
\end{aligned}$$

with $A = \mathbf{P}(\tau_+^{(a)} < \infty)$. Combining results (54) and (56) we attain

$$\mathbf{P}(M^{(a)} = y\Delta) = \mathbf{P}(\tau_+^{(a)} = \infty) \sum_{k=1}^{\infty} A^k \left(F_k^{(\mu^{(a)})}(y\Delta) - F_k^{(\mu^{(a)})}((y-1)\Delta) \right). \tag{57}$$

Next, we want to use Proposition 2 to determine the asymptotic behaviour of the sum on the right hand side of the latter equality. Therefore, let us first show that under the assumptions of Theorem 1,

$$Z^{(a)} \xrightarrow{w} Z^{(0)} \tag{58}$$

as $a \rightarrow 0$. It is known that

$$\mathbf{P}(\tau_+^{(a)} < \infty) \sim \mathbf{P}(\tau_+ < \infty) = 1. \tag{59}$$

Thus, as $a \rightarrow 0$,

$$\mathbf{P}(Z^{(a)} > x) = \frac{\mathbf{P}(\chi^{(a)} > x, \tau_+^{(a)} < \infty)}{\mathbf{P}(\tau_+^{(a)} < \infty)} \sim \mathbf{P}(\chi^{(a)} > x, \tau_+^{(a)} < \infty)$$

and, on the other hand, (1) and (59) imply that for every $R > 0$, as $a \rightarrow 0$,

$$\begin{aligned}
\mathbf{P}(\chi^{(a)} > x, R < \tau_+^{(a)} < \infty) &\leq \mathbf{P}(R < \tau_+^{(a)} < \infty) \\
&= \mathbf{P}(\tau_+^{(a)} < \infty) - \mathbf{P}(\tau_+^{(a)} \leq R) \sim \mathbf{P}(\tau_+ > R).
\end{aligned}$$

Further, by using (1) and the continuous mapping theorem,

$$\begin{aligned}
\mathbf{P}(\chi^{(a)} > x, \tau_+^{(a)} \leq R) &= \sum_{k=0}^{R-1} \mathbf{P}(S_{k+1}^{(a)} > x, \max_{1 \leq l \leq k} S_l^{(a)} \leq 0) \\
&\sim \sum_{k=0}^{R-1} \mathbf{P}(S_{k+1} > x, \max_{1 \leq l \leq k} S_l \leq 0) = \mathbf{P}(\chi > x, \tau_+ \leq R)
\end{aligned}$$

as $a \rightarrow 0$. Thus,

$$\limsup_{a \rightarrow 0} \mathbf{P}(Z^{(a)} > x) \leq \mathbf{P}(\chi > x, \tau_+ \leq R) + \mathbf{P}(\tau_+ > R)$$

and by letting $R \rightarrow \infty$ we conclude

$$\limsup_{a \rightarrow 0} \mathbf{P}(Z^{(a)} > x) \leq \mathbf{P}(\chi > x, \tau_+ < \infty) = \mathbf{P}(Z^{(0)} > x).$$

On the other side, the above calculations give

$$\liminf_{a \rightarrow 0} \mathbf{P}(Z^{(a)} > x) \geq \liminf_{a \rightarrow 0} \mathbf{P}(\chi^{(a)} > x, \tau_+^{(a)} \leq R) = \mathbf{P}(\chi > x, \tau_+ \leq R)$$

and by letting $R \rightarrow \infty$,

$$\liminf_{a \rightarrow 0} \mathbf{P}(Z^{(a)} > x) \geq \mathbf{P}(\chi > x, \tau_+ < \infty) = \mathbf{P}(Z^{(0)} > x).$$

This means that (58) holds under our assumptions.

Due to relation (16) of Chow [4] there exists a constant C such that

$$\mathbf{E}[(S_{\tau_+^{(a)}}^{(a)})^{1+\varepsilon}; \tau_+^{(a)} < \infty] \leq C \int_0^\infty \frac{u^{2+\varepsilon}}{\mathbf{E}[|S_{\tau_-^{(a)}}^{(a)}| \wedge u]} d\mathbf{P}(\max\{0, X^{(a)}\} < u)$$

Obviously,

$$\mathbf{E}[|S_{\tau_-^{(a)}}^{(a)}| \wedge u] \geq \mathbf{E}[|S_{\tau_-^{(a)}}^{(a)}| \wedge \Delta] \geq \mathbf{P}(S_1^{(a)} < 0) > 0$$

for all $u \geq \Delta$ and therefore

$$\mathbf{E}[(S_{\tau_+^{(a)}}^{(a)})^{1+\varepsilon}; \tau_+^{(a)} < \infty] \leq \frac{C}{\mathbf{P}(S_1 < 0)} \int_0^\infty u^{2+\varepsilon} d\mathbf{P}(\max\{0, X^{(a)}\} < u).$$

Hence, by virtue of (2),

$$\sup_{a \leq a_0} \mathbf{E}[(Z^{(a)})^{1+\varepsilon}] < \infty. \quad (60)$$

The convergence from (58) combined with (60) implies

$$\mu^{(a)} \rightarrow \mu^{(0)} \quad (61)$$

as $a \rightarrow 0$ by dominated convergence. It is known that for all $a > 0$ the stopping time $\tau_+^{(a)}$ is infinite with positive probability and that

$$\mathbf{P}(\tau_+^{(a)} = \infty) = 1/\mathbf{E}[\tau_-^{(a)}], \quad (62)$$

where $\tau_-^{(a)} = \min\{k \geq 1 : S_k^{(a)} \leq 0\}$ is the first weak descending ladder epoch. Totally analogously to (60), one can use (15) from Chow [4] to show that the existence of the second moment in assumption (2) implies $\sup_{a \leq a_0} \mathbf{E}[S_{\tau_-^{(a)}}^{(a)}] < \infty$.

Hence, one can use dominated convergence to show that

$$\mathbf{E}[S_{\tau_-^{(a)}}^{(a)}] \rightarrow \mathbf{E}[S_{\tau_-^{(0)}}]$$

as $a \rightarrow 0$. Thus, using (62), the known identity

$$\frac{\sigma^2}{2} = -\mu^{(0)} \mathbf{E}[S_{\tau_-^{(0)}}] \quad (63)$$

and Wald's identity imply that

$$\mathbf{P}(\tau_+^{(a)} = \infty) = \frac{1}{\mathbf{E}[\tau_-^{(a)}]} \sim \frac{a}{-\mathbf{E}[S_{\tau_-^{(0)}}]} \sim \frac{2a\mu^{(0)}}{\sigma^2}. \quad (64)$$

The assumption $ay = O(1)$ implies the existence of a constant C such that $y \leq C/a$. Therefore, by (64),

$$\mathbf{P}(\tau_+^{(a)} < \infty) \geq 1 - \frac{3C\mu^{(0)}}{\sigma^2 y} \quad (65)$$

for a small enough. Summing up the results from (61) and (65), this means that we can apply Proposition 2 for $I = \{\mu^{(a)} : 0 \leq a \leq a_0\}$ with $a_0 > 0$ small enough, $A_y = 1 - 3C\mu^{(0)}/(\sigma^2 y)$, $A = \mathbf{P}(\tau_+^{(a)} < \infty)$ and $s = 1 + \varepsilon$. Hence,

$$\sum_{k=1}^{\infty} A^k \left(F_k^{(\mu^{(a)})}(y\Delta) - F_k^{(\mu^{(a)})}((y-1)\Delta) \right) = \frac{(\lambda_{y\Delta}^{(a)}(A))^{-y-1}}{A\mu_y^{(a)}(\lambda_{y\Delta}^{(a)}(A))} + o(y^{-\min\{1, \varepsilon\}} \ln y) \quad (66)$$

and consequently, by combining equations (57), (66) and the fact that $1 - A = O(a)$, we attain

$$\mathbf{P}(M^{(a)} = y\Delta) = (1 - A) \frac{(\lambda_{y\Delta}^{(a)}(A))^{-y-1}}{A\mu_y^{(a)}(\lambda_{y\Delta}^{(a)}(A))} + o(ay^{-\min\{1, \varepsilon\}} \ln y). \quad (67)$$

Let us now determine $\lambda_{y\Delta}^{(a)}(A)$ and $\mu_y^{(a)}(\lambda_{y\Delta}^{(a)}(A))$. Write $\lambda_{y\Delta}$ and $\mu_y(\lambda_{y\Delta})$ instead of $\lambda_{y\Delta}^{(a)}(A)$ and $\mu_y^{(a)}(\lambda_{y\Delta}^{(a)}(A))$ respectively for abbreviation and put $\lambda_{y\Delta} = e^{\theta_{y\Delta}}$. According to the definition of $\lambda_{y\Delta}$, we want to find $\theta_{y\Delta}$ such that

$$\mathbf{E}[\exp\{\theta_{y\Delta} Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta] = \frac{1}{A}. \quad (68)$$

It turns out we don't need an exact solution for this equation and it is sufficient to determine θ_y such that

$$\mathbf{E}[\exp\{\theta_{y\Delta} Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta] = \frac{1}{A} + O(y^{-1-\varepsilon}). \quad (69)$$

By Taylor's formula,

$$\begin{aligned} & \mathbf{E}[\exp\{\theta_{y\Delta} Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta] \\ &= 1 + \frac{\theta_{y\Delta} \mu^{(a)}}{\Delta} - \mathbf{P}(Z^{(a)} > y\Delta) - \frac{\theta_{y\Delta}}{\Delta} \mathbf{E}[Z^{(a)}; Z^{(a)} > y\Delta] \\ & \quad + \frac{\theta_{y\Delta}^2}{2\Delta^2} \mathbf{E}[(Z^{(a)})^2 \exp\{\gamma \theta_{y\Delta} Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta] \end{aligned}$$

with some random $\gamma \in (-\infty, 1]$. We restrict ourselves to $\theta_{y\Delta}$ such that $\theta_{y\Delta} = O(1/y)$. Then, (60) implies

$$\mathbf{P}(Z^{(a)} > y\Delta) + \frac{\theta_{y\Delta}}{\Delta} \mathbf{E}[Z^{(a)}; Z^{(a)} > y\Delta] = O(y^{-1-\varepsilon})$$

and

$$\begin{aligned} & \frac{\theta_{y\Delta}^2}{2\Delta^2} \mathbf{E}[(Z^{(a)})^2 \exp\{\gamma \theta_{y\Delta} Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta] \\ &= O\left(\theta_{y\Delta}^2 E[(Z^{(a)})^2; Z^{(a)} \leq y\Delta]\right) = O(y^{-1-\varepsilon}). \end{aligned}$$

This means that to find θ_y that suffices (69), it is sufficient to choose θ_y such that

$$1 + \frac{\theta_{y\Delta} \mu^{(a)}}{\Delta} = \frac{1}{A} + O(y^{-1-\varepsilon})$$

or

$$\theta_{y\Delta} = \frac{(1-A)\Delta}{A\mu^{(a)}} + O(y^{-1-\varepsilon}).$$

Consequently,

$$\lambda_{y\Delta} = \exp\left\{\frac{(1-A)\Delta}{A\mu^{(a)}} + O(y^{-1-\varepsilon})\right\}. \quad (70)$$

Further,

$$\begin{aligned}\mu_{y\Delta}^{(a)}(\lambda_{y\Delta}) &= \sum_{k=1}^y k f_k^{(a)} \lambda_{y\Delta}^{k-1} = \frac{1}{\Delta \lambda_{y\Delta}} \mathbf{E}[Z^{(a)} \exp\{\theta_{y\Delta} Z^{(a)} / \Delta\}; Z^{(a)} \leq y\Delta] \\ &= \frac{1}{\Delta \lambda_{y\Delta}} \left\{ \mathbf{E}[Z^{(a)}; Z^{(a)} \leq y\Delta] + \frac{\theta_{y\Delta}}{\Delta} \mathbf{E}[(Z^{(a)})^2 \exp\{\tilde{\gamma} \theta_{y\Delta} Z^{(a)} / \Delta\}; Z^{(a)} \leq y\Delta] \right\}\end{aligned}$$

for some random $\tilde{\gamma} \in (-\infty, 1]$. For all $\theta_{y\Delta} = O(1/y)$ the result (60) gives

$$\mathbf{E}[(Z^{(a)})^2 \exp\{\tilde{\gamma} \theta_{y\Delta} Z^{(a)} / \Delta\}; Z^{(a)} \leq y\Delta] = O(y^{1-\varepsilon})$$

and

$$\mathbf{E}[Z^{(a)}; Z^{(a)} \leq y\Delta] = \mu^{(a)} + O(y^{-\varepsilon}).$$

Consequently,

$$\mu_{y\Delta}^{(a)}(\lambda_{y\Delta}) = \frac{\mu^{(a)}}{\Delta \lambda_{y\Delta}} + O(y^{-\varepsilon}). \quad (71)$$

Plugging the results from (70) and (71) into the right hand side of (67), we obtain by regarding $1 - A = O(a)$,

$$\begin{aligned}\mathbf{P}(M^{(a)} = y\Delta) &= \frac{(1-A)\Delta}{A\mu^{(a)} + O(y^{-\varepsilon})} \exp\left\{-\frac{(1-A)y\Delta}{A\mu^{(a)}} + O(y^{-\varepsilon})\right\} + o(ay^{-\min\{1,\varepsilon\}} \ln y) \\ &= \frac{(1-A)\Delta}{A\mu^{(a)} + O(y^{-\varepsilon})} \exp\left\{-\frac{(1-A)y\Delta}{A\mu^{(a)}}\right\} + o(ay^{-\min\{1,\varepsilon\}} \ln y) \\ &= \frac{(1-A)\Delta}{A\mu^{(a)}} \exp\left\{-\frac{(1-A)y\Delta}{A\mu^{(a)}}\right\} + o(ay^{-\min\{1,\varepsilon\}} \ln y) + O(ay^{-\varepsilon})\end{aligned} \quad (72)$$

uniformly for all y such that $ay = O(1)$ as $a \rightarrow 0$. Here, we applied Taylor's formula in the last line. As a consequence of (59), (61) and (64),

$$\frac{1-A}{A\mu^{(a)}} = \frac{2a}{\sigma^2} + o(a)$$

and hence, by plugging this result into (72), we finally obtain

$$\mathbf{P}(M^{(a)} = y\Delta) \sim \frac{2a\Delta}{\sigma^2} \exp\left\{-\frac{2ay\Delta}{\sigma^2}\right\}$$

uniformly for all y such that $y \rightarrow \infty$ and $ya = O(1)$ as $a \rightarrow 0$.

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